Sampled-data stabilization of feedforward dynamics with Lyapunov cross-term

Mattia Mattioni, Salvatore Monaco and Dorothée Normand-Cyrot

***Abstract*— The paper addresses the problem of preserving the stabilizing performances of a continuous-time feedback under sampling. This is discussed with reference to a two-block feedforward dynamics for which the continuous-time design is based on the construction of a Lyapunov cross term.**

***Index Terms*— Lyapunov methods, Algebraic/geometric methods, Stability of nonlinear systems**

I. INTRODUCTION

This paper addresses the problem of designing a sampled-data state feedback for stabilizing a continuous-time (CT) two-block cascade system admitting a feedforward structure. In this context, we assume that the measures are available at sampling instants and that the control is constant over time intervals of length δ, the sampling period. We employ a passivity-based approach exploiting the construction of a Lyapunov function with a cross-term. Three sampled-data (SD) stabilizing strategies are discussed while a working example illustrates the performances.

The paper is organized as follows. In Section II the problem is formulated. In Section III preliminaries on sampled-data equivalent models to feedforward cascade are reported. Section IV discusses three different sampled-data stabilizing strategies. An academic example is treated and simulations are depicted in Section V. Some conclusions are set in Section VI.

Notations: All the functions and vector fields defining the dynamics are assumed smooth over the respective definition spaces. M_U denotes the space of measurable and locally bounded functions $u : \mathbb{R}^+ \to U$, with $U \subseteq \mathbb{R}$ and by M_U^I the space of measurable and locally bounded functions $u : I \to U$, with $I \subset \mathbb{R}$. $\mathscr{U}^\delta \subseteq M_U$ denotes the set of piecewise constant functions over time intervals of length $\delta \in]0, T^*[$, a finite time interval; i.e. $\mathscr{U}^\delta = \{u \in M_U \text{ s.t. } u(t) = u_k,\ \forall t \in [k\delta, (k+1)\delta[\text{ and } k \geq 0\}$. Given a vector field on $\mathbb{R}^n$, L_f denotes the associated Lie derivative operator, $L_f = \sum_{i=1}^n f_i(\cdot)\frac{\partial}{\partial x_i}$. e^{L_f}(or e^f, when no confusion arises) denotes the associated Lie series operator, $e^f := I_d + \sum_{i\geq 1} \frac{L_f^i}{i!}$ where I_d indicates the identity operator. Given a smooth mapping $h : \mathbb{R}^\ltimes \to \mathbb{R}$, one has $h(e^f I_d\big|_x) = e^f h(x)$. Given two vector fields f, g over $\mathbb{R}^n$, their Lie bracket is defined as $ad_f g := [f, g] := [L_f, L_g] := L_f \circ L_g - L_g \circ L_f$ and, iteratively, $ad_f^i g := [f, ad_f^{i-1} g]$, with $ad_f^0 g := g$.

*This work was supported by *Programme International Cooperation Scientifique* (PICS, CNRS ST2I) for collaboration between Italy and France.

M. Mattioni is with both L2S (CNRS-CentraleSupelec-Univ. Paris-Sud) and DIAG (Università di Roma "La Sapienza"). mattia.mattioni@l2s.centralesupelec.fr

S.Monaco is with Dipartimento di Ingegneria Informatica, Automatica e Gestionale, DIAG (Università di Roma "La Sapienza") via Ariosto 25, 00185 Roma, Italy. salvatore.monaco@uniroma1.it

D. Normand-Cyrot is with Laboratoire des Signaux et Systèmes (L2S, UMR CNRS 8506) CNRS - Centrale-Supelec; 3, rue Joliot Curie, 91192, Gif-sur-Yvette, France cyrot@l2s.centralesupelec.fr

II. PROBLEM FORMULATION

Consider the two-block continuous-time feedforward system

$$\Sigma_c : \begin{cases} \dot{z} = f(z) + \varphi(z,\xi) + g(z,\xi)u, & z \in \mathbb{R}^n \quad (1) \\ \dot{\xi} = a(\xi) + b(\xi)u, & \xi \in \mathbb{R}^m,\ u \in \mathbb{R} \quad (2) \end{cases}$$

with equilibrium at the origin. Denote by Σ_0 the cascade structure deduced from Σ_c when $u = 0$. The following standing assumptions are set.

Assumption 2.1: $\dot{z} = f(z)$ is globally stable (GS), with radially unbounded and locally quadratic Lyapunov function $W(z)$ ($L_f W(z) \leq 0$ for all z). $\dot{\xi} = a(\xi)$ is globally asymptotically stable (GAS) and locally exponentially stable (LES) with radially unbounded and locally quadratic storage function $U(\xi)$ verifying $L_a U(\xi) < 0$ for any ξ.

Assumption 2.2 (linear growth of the function $\varphi(z,\xi)$): there exist two class $\mathscr{K}$-functions* $\gamma_1(\cdot)$ and $\gamma_2(\cdot)$ such that

$$||\varphi(z,\xi)|| \leq \gamma_1(||\xi||)||z|| + \gamma_2(||\xi||).$$

Assumption 2.3 (growth of the Lyapunov function $W(z)$): there exist real constants c and M such that, for $\|z\| > M$

$$\|\frac{\partial W}{\partial z}\|\|z\| \leq cW(z).$$

It is well known that whenever the above assumptions are verified the equilibrium of Σ_0 is Globally Stable (GS). Nevertheless, the sum of the functions $W(z)$ and $U(\xi)$ cannot provide, provide a Lyapunov function for the interconnected system Σ_0. Among other approaches (see [1]), a Lyapunov function $V_0(z,\xi)$ for the complete system has been given in [2] via the construction of a cross term $\Psi(z,\xi)$. The following result states the existence of a suitable Lyapunov function V_0 which is non-increasing along the trajectories of Σ_0 whenever Assumptions 2.1 to 2.3 are verified.

Theorem 2.1 ([2]): Under Assumptions 2.1, 2.2 and 2.3, the equilibrium $(z,\xi) = (0,0)$ of Σ_0 is GS with radially unbounded and positive definite Lyapunov function $V_0(z,\xi) = W(z) + U(\xi) + \Psi(z,\xi)$. The cross term $\Psi(z,\xi)$ is continuous on $\mathbb{R}^n \times \mathbb{R}^m$ and takes the form

$$\Psi(z,\xi) = \int_0^\infty \frac{\partial W}{\partial z}\big|_{\bar{z}(s,z,\xi)} \varphi(\bar{z}(s,z,\xi), \bar{\xi}(s,\xi))ds$$

where $(\bar{z}(s,z,\xi),\bar{\xi}(s,\xi))$ denote the trajectories of Σ_0 at time s with initial condition (z,ξ).

Accordingly, GAS of the equilibrium of Σ_c under the passivity-based controller (PBC) $u = -\mathrm{L}_{\bar{g}}V_0$ with $\bar{g} = col[g;b]$ immediately follows whenever Σ_c, with output $y = h(z,\xi) = \mathrm{L}_{\bar{g}}V_0$, is zero state detectable (see for example [3]).

Definition 2.1 (ZSD): Let Σ_c have output $y = h(z,\xi)$ and $Z \subset \mathbb{R}^n$ be the largest positively invariant set contained in $\{x \in \mathbb{R}^n \mid y = h(z,\xi) = 0\}$; Σ_c with output $y = h(z,\xi)$ is said Zero-State-Detectable (ZSD) if the origin is GAS conditionally to Z.

The present paper discusses the preservation of these results under sampling. The following properties are proven.

- the feedforward structure of Σ_c (as well as the one of Σ_0) is maintained by the sampled-data equivalent dynamics Σ^δ (and Σ_0^δ when $u = 0$);
- the Lyapunov function V_0 constructed for Σ_0 is a Lyapunov function for Σ_0^δ;
- global asymptotic stabilization under sampled-data state feedback can be achieved whenever the ZSD property is verified in continuous-time with respect to $y = \mathrm{L}_{\bar{g}}V_0$.

For this last item, three different sampled-data stabilizing strategies are discussed.

- *input Lyapunov matching* (I-LM) strategy, as proposed in [4];
- SD predictive output feedback passivity with respect to the *time-average output* associated to $\mathrm{L}_{\bar{g}}V_0$ as discussed in [5] (see also the use of this time average output in [6]);
- SD u-average passivity based control with respect to the *u-average output* associated to the system Σ^δ as proposed in [5]-[7].

In the following, V_0 is assumed at least C^2 (i.e., the first and second derivatives exist and are continuous). According to [2] considering smooth vector fields defining Σ_c, this is not restrictive.

III. Sampled-data equivalent models

Let the system Σ_c be rewritten over $\mathbb{R}^n \times \mathbb{R}^m$ as

$$\Sigma_c: \quad \dot{\bar{x}}(t) = \bar{f}(\bar{x}) + u(t)\bar{g}(\bar{x})$$

with $\bar{x} = (z,\xi)$, $\bar{f} = col[f(z) + \varphi(z,\xi), a(\xi)]$ and $\bar{g} = col[g(z,\xi); b(\xi)]$. Setting $u(t) \in \mathscr{U}^\delta$, the SD equivalent model Σ^δ to Σ_c is defined in the form of a map as

$$\Sigma^\delta: \quad \bar{x}_{k+1} = e^{\delta(L_{\bar{f}} + u_k L_{\bar{g}})}\bar{x}_k$$

with $\bar{x}_k = \bar{x}(t = k\delta)$ for any $k \geq 0$. According to [8], Σ^δ admits an $(\bar{F}_0^\delta, \bar{G}^\delta)$ - representation described as two coupled difference and differential equations

$$\bar{x}^+ = \bar{F}_0^\delta(\bar{x}), \qquad \bar{x}^+(0) = \bar{x}^+ \tag{3a}$$

$$\frac{\partial \bar{x}^+(u)}{\partial u} = \bar{G}^\delta(\bar{x}^+(u), u) \tag{3b}$$

with $\bar{x}^+(u) = e^{\delta(L_{\bar{f}} + uL_{\bar{g}})}\bar{x}$ for any $u \in \mathscr{U}^\delta$ and by definition

$$\bar{F}_0^\delta(\bar{x}) = e^{\delta L_{\bar{f}}}\bar{x}; \quad \bar{G}^\delta(\bar{x}^+(u), u) = \int_0^\delta e^{-s ad_{\bar{f}+u\bar{g}}}\bar{g}(\bar{x}^+(u))\mathrm{d}s.$$

Accordingly, one verifies that by definition $\bar{G}^\delta(\bar{x}^+(u), u) = \frac{\partial}{\partial u} e^{\delta(L_{\bar{f}} + uL_{\bar{g}})}(\bar{x})$ and one computes for any given pair $(\bar{x}_k, u_k)$

$$\bar{x}_{k+1} = \bar{x}_k^+(u_k) = \bar{x}_k^+(0) + \int_0^{u_k} L_{\bar{G}^\delta(\cdot,v)}(\bar{x}_k^+(v))\mathrm{d}v$$

with $\bar{x}_k^+(0) = \bar{F}_0^\delta(\bar{x}_k)$. In this context, given any C^1 function $S: \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}$, $S(\bar{x}_{k+1})$ can be written around $S(\bar{x}_k^+(0))$ in the integral form below

$$S(\bar{x}_k^+(u_k)) = S(\bar{x}_k^+(0)) + \int_0^{u_k} L_{\bar{G}^\delta(\cdot,v)} S(\bar{x}_k^+(v))\mathrm{d}v$$

which will be useful in the sequel.

A. The sampled-data feedforward structure

It is a matter of computation to verify the following.

Lemma 3.1: Given Σ_c, its sampled-data equivalent Σ^δ preserves the feedforward structure and can described by two cascade dynamics in their differential difference form; i.e.

$$z^+ = f^\delta(z) + \varphi^\delta(z,\xi), \quad z^+(0) = z^+ \tag{4a}$$

$$\frac{\partial z^+(u)}{\partial u} = G^\delta(z^+(u), \xi^+(u), u) \tag{4b}$$

and

$$\xi^+ = a^\delta(\xi), \quad \xi^+(0) = \xi^+ \tag{5a}$$

$$\frac{\partial \xi^+(u)}{\partial u} = B^\delta(\xi^+(u), u) \tag{5b}$$

with

$$f^\delta(z) = e^{\delta L_f}z; \quad \varphi^\delta(z,\xi) = e^{\delta L_{\bar{f}}}z - e^{\delta L_f}z$$

$$G^\delta(z^+(u), \xi^+(u), u) = \int_0^\delta e^{-s ad_{\bar{f}+u\bar{g}}} g(z^+(u))\mathrm{d}s$$

$$a^\delta(\xi) = e^{\delta L_a}\xi; \quad B^\delta(\xi^+(u), u) = \int_0^\delta e^{-s ad_{a+ub}} b(\xi^+(u))\mathrm{d}s.$$

B. Lyapunov function with a cross-term

Let Σ_0 verify Assumptions 2.1 to 2.3, then the candidate Lyapunov function $V_0(z,\xi)$ is in the form

$$V_0(z,\xi) = W(z) + \Psi(z,\xi) + U(\xi) \tag{6}$$

with cross-term $\Psi(z,\xi)$ computed to satisfy

$$\dot{\Psi}(z,\xi) = -L_{\varphi(z,\xi)}W(z). \tag{7}$$

(7) guarantees that V_0 is non-increasing along the trajectories of Σ_0 since

$$\dot{V}_0(z,\xi) = L_{f(z)}W(z) + L_{\varphi(z,\xi)}W(z) + \dot{\Psi}(z,\xi) + L_a U(\xi) \leq 0$$

by construction.
The following result is an immediate consequence of the definition of the function V_0.

Lemma 3.2: Given Σ_c satisfying Assumptions 2.1 to 2.3 with GS equilibrium when $u = 0$. Then $V_0(z,\xi)$ as in (6) verifying (7) is a Lyapunov function for Σ_0^δ; i.e., along the trajectories of Σ_0^δ, one verifies for any $k \geq 0$ and any pair $(z_k, \xi_k))$

$$\Delta_k V_0 := V_0(z_{k+1}, \xi_{k+1}) - V_0(z_k, \xi_k) \leq 0$$

where $\Delta_k V_0(z,\xi) = \Delta_k U(\xi_k) + \int_0^\delta L_f W(\bar{z}(s, z_k, \xi_k))\mathrm{d}s$ and $\bar{z}(s, z_k, \xi_k)$ denotes the solution of (1) at time $s + k\delta$ with initial condition (z_k, ξ_k) at $t = k\delta$.

The result is an immediate consequence of time integration of the continuous-time condition $\dot{V}_0 \leq 0$ over each time interval of length δ; accordingly, one gets

$$\Delta_k V_0(z,\xi) := \int_0^\delta \dot{V}_0(\bar{z}(s, z_k, \xi_k), \bar{\xi}(s, \xi_k)\mathrm{d}s \leq 0.$$

It immediately follows that the cross term $\Psi(z,x)$ satisfies (along the sampled-data trajectories) the equality

$$\Delta_k \Psi(z,\xi) = -\int_0^\delta L_{\varphi(\bar{z}(s,z_k,\xi_k), \bar{\xi}(s,\xi_k))} W(\bar{z}(s, z_k, \xi_k))\mathrm{d}s.$$

Remark 3.1: It is important to note that the Lyapunov function for the continuous-time system Σ_c is still a Lyapunov function for the sampled-data one Σ^δ. A different Lyapunov function V_0^δ can be built for Σ_0^δ by directly looking for a cross term $\Psi^\delta(z,\xi)$ to satisfy the equality

$$\Delta\Psi^\delta(z,\xi) = -W(f^\delta(z) + \varphi^\delta(z,\xi)) + W(f^\delta(z)).$$

Arguing so, one cancels all the cross terms which appear when computing

$$\Delta V_0^\delta = \Delta U(\xi) + W(f^\delta(z) + \varphi^\delta(z,\xi)) - W(z) + \Delta\Psi^\delta(z,\xi)$$

with $V_0^\delta(z,\xi) = W(z) + \Psi^\delta(z,\xi) + U(\xi)$. By construction, one gets the required condition

$$\Delta V_0^\delta(z,\xi) = \Delta U(\xi) + W(f^\delta(z)) - W(z) \leq 0$$

along the trajectories of Σ_0^δ. The conditions under which $\Psi^\delta(z,\xi)$ exists (or can be deduced from $\Psi(z,\xi)$) deserve further investigation.

IV. SD ASYMPTOTIC STABILIZATION

Three different SD strategies for achieving asymptotic stabilization are discussed. These three controllers are characterized by their series expansions in powers of δ around the continuous-time stabilizing feedback; i.e. the sampled-data control u is defined by its series expansion

$$u = u_0 + \sum_{i>0} \frac{\delta^i}{(i+1)!} u_i \tag{8}$$

with $u_0 = u_c$ and where each u_i denotes an additional term of order i (making reference to the corresponding coefficient δ^i) in the series expansion.

The following example is worked out in the sequel.

Example: Consider the cascade system

$$\dot{z} = \xi + \xi^3 - \xi^2 u; \quad \dot{\xi} = -\xi + u \tag{9}$$

which verifies Assumptions 2.1 to 2.3 with $W(z) = \frac{1}{2}z^2$ and $U(\xi) = \frac{1}{2}\xi^2$. The cross term $\Psi(z,\xi) = \frac{1}{2}(z + \xi + \frac{\xi^3}{3})^2 - \frac{1}{2}z^2$ verifies $\dot{\Psi}(z,\xi) = -z(\xi + \xi^3)$. (9) is passive with output $y_c = \frac{1}{3}\xi^3 + 2\xi + z$. The CT feedback is provided by $u_c = -y_c$.

Setting $u \in \mathscr{U}^\delta$, the sampled-data equivalent model of (9) can be exactly computed (i.e. Σ_c is integrable when $u \in \mathscr{U}^\delta$) with (F_0^δ, G^δ)-representation as below

$$\begin{aligned}
z^+ &= z + \xi - \frac{e^{-3\delta}}{3}\xi^3 + \frac{1}{3}\xi^3 - e^{-\delta}\xi \\
\frac{\partial z^+(u)}{\partial u} &= \delta + e^{-\delta} - 1 + (e^{-\delta} - 1)[\xi^+(u)]^2 \\
\xi^+ &= e^{-\delta}\xi, \quad \frac{\partial \xi^+(u)}{\partial u} = 1 - e^{-\delta}.
\end{aligned}$$

A. Input-Lyapunov Matching SD-controller

The driving idea is to preserve, through piecewise constant controls and at the sampling instants, the properties of some function involved in the continuous-time design. Such methodology has been developed by two of the authors in several contributions, starting from the matching of some real (or dummy) output mapping to preserve the minimum phase criterion [9] up to the matching of Hamiltonian functions in the context of optimality [10]. In particular, such approach is constructive when matching, at the sampling instants, the closed loop behaviour of a Lyapunov function so getting a sort of Lyapunov based design under sampling. This strategy is developed in [4] in the context of backstepping design under sampling.

In the present paper, we assume the existence of a continuous-time state feedback $u_c(\bar{x})$ achieving global asymptotic stability of the closed-loop equilibrium of Σ_c with Lyapunov $V_0(\bar{x})$. The idea is to find a SD state feedback $u^{ilm} = u^{ilm}(\bar{x}_k)$ so that along the sampled-data trajectories, the closed loop evolution of $V_0(\bar{x})$ is matched at the sampling instants; i.e. the control u^{ilm} satisfies, at each sampling instant $k \geq 0$, and for each pair $(\bar{x}_k, u_k)$ the following equality

$$e^{\delta(L_{\bar{f}} + u_k L_{\bar{g}})} V_0(\bar{x})\Big|_{\bar{x}_k} - V_0(\bar{x}_k) = \int_0^\delta \dot{V}_0(e^{s(L_{\bar{f}} + u_c L_{\bar{g}})}(\bar{x})\Big|_{\bar{x}_k})ds \tag{10}$$

when $\bar{x}_k = \bar{x}(t = k\delta)$ and $u_k = u^{ilm}(x_k)$. The following result holds true.

Theorem 4.1: Let Σ_c verify Assumptions 2.1, 2.2 and 2.3 with control Lyapunov function $V_0(z,\xi)$ and be ZSD with output mapping $y_c = \mathrm{L}_{\bar{g}} V_0(z,\xi)$. Let the CT feedback $u_c = -K\mathrm{L}_{\bar{g}} V_0(z,\xi)$ $(K > 0)$ achieve GAS of the closed-loop equilibrium. Then, there exists a SD state feedback u^{ilm} in the form (8) ensuring Input Lyapunov Matching of $V_0(z,\xi)$ so yielding GAS of the closed-loop equilibrium of Σ^δ.

Sketch of proof: We refer to [4] for the details of proof and constructive aspects. Briefly, the existence of a unique solution to (10) in the form (8) is deduced from the Implicit Function Theorem provided $L_{\bar{g}} V_0(\bar{x}) \neq 0$ for any $\bar{x} \neq 0$. The solution u^{ilm} to (10) can be iteratively computed by substituting (8) into (10) and equating the terms with the same power of δ. For the first terms one computes

$$u_0^{ilm} = u_c\big|_{t=k\delta}, \quad u_1^{ilm} = \dot{u}_c\big|_{t=k\delta} = (L_{\bar{f}} + u_c L_{\bar{g}})u_c\big|_{t=k\delta}$$
$$u_2^{ilm} = \ddot{u}_c\big|_{t=k\delta} + \frac{\dot{u}_c\big|_{t=k\delta}}{2L_{\bar{g}}V_0(\bar{x}_k)} L_{[\bar{f},\bar{g}]}V_0(\bar{x}_k) \tag{11}$$

so recovering the CT solution for $\delta = 0$.

The following inequality holds at the sampling instants and is inherited from the I-LM property

$$\Delta_k V_0 \leq -K \int_0^\delta \| e^{s(L_{\bar{f}}+u_c L_{\bar{g}})} \mathrm{L}_{\bar{g}} V_0(\bar{x})\big|_{\bar{x}_k} \|^2 \mathrm{d}s \leq$$
$$-K \| \int_0^\delta e^{s(L_{\bar{f}}+u_c L_{\bar{g}})} \mathrm{L}_{\bar{g}} V_0(\bar{x})\big|_{\bar{x}_k} \mathrm{d}s \|^2. \tag{12}$$

Remark 4.1: When designing u^{ilm} via matching of the behavior of the control Lyapunov function V_0, the SD closed-loop trajectories inherit the continuous-time properties associated to V_0. This is of particular interest when robustness or optimality are guaranteed by V_0 in continuous time and so preserve under sampling by construction of u^{ilm}.

Example (cont'd): Consider again the case of (9). The terms (11) specialise as follows

$$u_0^{ilm} = -\frac{1}{3}\xi^3 - 2\xi - z; \quad u_1^{ilm} = \frac{2}{3}\xi^3 + 5\xi + 2z$$
$$u_2^{ilm} = Q(z,\xi)(15z^2 - 10z\xi^3 - 66z\xi - \frac{5\xi^6}{3} - 22\xi^4 - \frac{141\xi^2}{2})$$

with $Q(z,\xi) = \frac{1}{6}(\xi^3 + 6\xi + 3z)^{-1}$.

Two further strategies based are detailed below by re-designing output mappings so guaranteeing some dissipativity properties of the equivalent SD dynamics Σ^δ.

B. SD δ-predicted output feedback

It is well known that the passivity properties are lost under sampling as discussed in [11] and [6]. In [5], the authors show that, given $y = h(\bar{x})$ ensuring passivity of Σ_c with storage function V_0, one can redesign the output mapping so that a certain dissipativity inequality is verified under sampling. This is performed by introducing the so-called *δ-average predicted output* deduced from $h(\bar{x})$ via time-averaging between to successive sampling instants. Namely, given $y = h(\bar{x})$, one sets

$$y^\delta(\bar{x}_k, u_k) = h^\delta(\bar{x}_k, u_k) = \frac{1}{\delta}\int_0^\delta e^{s(L_{\bar{f}}+u_k L_{\bar{g}})} h(\bar{x})\big|_{\bar{x}_k} ds. \tag{13}$$

The next result specifies Theorem 3.1 and Theorem 3.2 in [5] to the present context.

Theorem 4.2: Let Σ_c verify Assumptions 2.1, 2.2 and 2.3 so that Σ_c with output $y = h(z,\xi) = \mathrm{L}_{\bar{g}} V_0(z,\xi)$ is passive with storage function $V_0(z,\xi)$. Then, the SD equivalent model Σ^δ is passive with output mapping $y^\delta = h^\delta(\bar{x},u)$ in (13) and storage function $V_0(z,\xi)$. Furthermore, the controller u^δ, solution of the equality

$$u^\delta + Kh^\delta(z,\xi,u^\delta) = 0, \quad K > 0 \tag{14}$$

ensures GAS of the closed-loop equilibrium provided that the continuous-time system Σ_c with mapping $y = h(z,\xi)$ is ZSD.

Sketch of proof: The proof is constructive and the SD δ-predicted output feedback comes in the form of a series expansion in powers of δ because so is $h^\delta(z,\xi,u)$

$$h^\delta(z,\xi,u) = h(z,\xi) + \sum_{i\geq 1} \frac{\delta^i}{(i+1)!}(L_{\bar{f}} + uL_{\bar{g}})h\big|_{\bar{x}}.$$

Passivity follows from the direct time integration of the passivity inequality satisfied by Σ_c with output $y = h(z,\xi) = \mathrm{L}_{\bar{g}} V_0(z,\xi)$ and storage function $V_0(z,\xi)$; i.e.

$$\dot{V}_0(z(t),\xi(t)) \leq u(t)\mathrm{L}_{\bar{g}} V_0(z(t),\xi(t)).$$

Setting $u(t) \in \mathscr{U}^\delta$ and integrating over $[k\delta,(k+1)\delta[$, one gets for any triplet (z_k,ξ_k,u_k)

$$\Delta_k V_0(z_k,\xi_k) \leq u_k \int_0^\delta e^{s(L_{\bar{f}}+u_k L_{\bar{g}})} \mathrm{L}_{\bar{g}} V_0(z,\xi)\big|_{\bar{x}_k} ds, \ k \geq 0$$

which can be seen as a passivity inequality with respect to $y^\delta = h^\delta(z_k,\xi_k,u_k)$; i.e.

$$\Delta_k V_0(z_k,\xi_k) \leq \delta u_k h^\delta(z_k,\xi_k,u_k).$$

The state feedback implicitly defined as $u^\delta = -Kh^\delta(z_k,\xi_k,u^\delta)$ achieves GAS of closed loop equilibrium and yields the inequality

$$\Delta_k V_0(z,\xi) \leq -\delta K \|y^\delta(z_k,\xi_k,u^\delta\|^2, \quad K \geq 0. \tag{15}$$

For the first terms, one computes

$$u_0^\delta = u_c\big|_{t=k\delta}, \quad u_1^\delta = \dot{u}_c\big|_{t=k\delta}$$
$$u_2^\delta = \ddot{u}_c\big|_{t=k\delta} - \frac{1}{2}\dot{u}_c\big|_{t=k\delta} L_{\bar{g}}^2 V_0(\bar{x}_k).$$

Remark 4.2: When comparing inequalities (12) and (15), both controllers ensure stabilization under sampling. Nevertheless, while in the I-LM approach the passive output is integrated along the continuous-time closed loop trajectories of Σ_c, in the second case, the integral is performed along the sampled-data closed loop trajectories.

Remark 4.3: By exploiting the $(F_0^\delta, \bar{G}^\delta)$-form in (3), the control (14) rewrites as

$$u^\delta = -K(T_1^\delta(\bar{x},u^\delta))^{-1} T_2^\delta(\bar{x}) \tag{16}$$

with

$$T_1^\delta(\bar{x},u^\delta) = \delta + \int_0^\delta \int_0^1 L_{\bar{G}^s(\cdot,\theta u^\delta)} h(\bar{x}^+(\theta u^\delta)) d\theta ds \tag{17}$$
$$T_2(x) = \int_0^\delta h(e^{sL_{\bar{f}}}\mathrm{Id}\big|_{\bar{x}}) ds. \tag{18}$$

Example (cont'd): The δ- predicted output is exactly given by

$$\delta h^\delta = \xi - u + \delta u + \delta z + \delta\xi + \frac{\delta^2}{2}u + \frac{\delta}{3}\xi^3 + ue^{-\delta} - \xi e^{-\delta}$$

so that u^δ is given by

$$u^\delta = -2D^\delta(\delta)(\xi + \delta z + \delta\xi + \frac{\delta}{3}\xi^3 - \xi e^{-\delta})$$

and $D^\delta(\delta) = (4\delta + 2e^{-\delta} + \delta^2 - 2)^{-1}$.

C. SD u-average passive output feedback

In the previous subsection we have shown that passivity is preserved under sampling when redesigning the output mapping as directly dependent on the input variable. The need of a direct throughput link is a typical requirement to give sense to passivity in discrete time. The notion of *u-average passivity* has been introduced in [7] to overcome the corresponding difficulty, . The following definition is recalled.

Definition 4.1: Σ^δ in (3) with output $H(\bar{x},u)$ is *u-average passive* if there exists a storage function $S(\cdot):\mathbb{R}^{n+m}\to\mathbb{R}_{\geq 0}$ such that $S(\bar{x}_{k+1})-S(\bar{x}_k)\leq u_k H^{av}(\bar{x}_k,u_k)$ where $H^{av}(\bar{x},u)$ denotes the *u-average output mapping* associated to $H(\bar{x},u)$ and defined by

$$H^{av}(\bar{x},u):=\frac{1}{u}\int_0^u H(\bar{x}^+(v),v)\mathrm{d}v.$$

Accordingly, given Σ^δ in (3) and the Lyapunov function $V_0(\cdot)$, one considers first the function $H(\cdot,u)=\mathrm{L}_{\bar{G}^\delta(\cdot,u)}V_0(\cdot)$ and then the associated *u-average output mapping* to set

$$\begin{aligned}\delta H^\delta_{av}(\bar{x},u):=&\frac{1}{u}\int_0^u \mathrm{L}_{\bar{G}^\delta(\cdot,v)}V_0(\bar{x}^+(v))\mathrm{d}v \\ =&\int_0^1 \mathrm{L}_{\bar{G}^\delta(\cdot,\theta u)}V_0(\bar{x}^+(\theta u))\mathrm{d}\theta\end{aligned}\qquad(19)$$

On these bases, a result in [7] is specified to the present context.

Theorem 4.3: Let Σ_c verify Assumptions 2.1, 2.2 and 2.3 so that it is passive with respect to the dummy output $\mathrm{L}_{\bar{g}}V_0(\bar{x})$ with storage function $V_0(\bar{x})$. Then Σ^δ is u-average passive with respect to the dummy output function $\mathrm{L}_{\bar{G}^\delta(\cdot,u)}V_0(\bar{x})$ with the same storage function $V_0(\bar{x})$. Furthermore, the SD controller u^{av} defined as the implicit solution of

$$u^{av}+K\delta H^\delta_{av}(\bar{x},u^{av})=0,\quad K>0\qquad(20)$$

makes the closed-loop equilibrium of Σ^δ GAS provided that the dummy output $\mathrm{L}_{\bar{g}}V_0(\bar{x})$ is ZSD for Σ_c.

Sktech of proof: The proof is still constructive and exploits the $(\bar{F}^\delta_0,\bar{G}^\delta)$-structure. First, one rewrites the increment of $V_0(\bar{x}_k)$ along the trajectories of Σ^δ as

$$\Delta_k V_0(x_k)=V_0(\bar{F}^\delta_0(\bar{x}_k))-V_0(\bar{x}_k)+\int_0^{u_k}\mathrm{L}_{\bar{G}^\delta(\cdot,v)}V_0(\bar{x}^+_k(v))\mathrm{d}v$$

with $V_0(\bar{F}^\delta_0(\bar{x}_k))-V_0(\bar{x}_k)\leq 0$ by construction of $V_0(\bar{x}_k)$. Thus, one gets

$$\Delta_k V_0(\bar{x}_k)\leq\int_0^{u_k}\mathrm{L}_{\bar{G}^\delta(\cdot,v)}V_0(\bar{x}^+_k(v))\mathrm{d}v.$$

so emphasizing u-average passivity with respect to $\mathrm{L}_{\bar{G}^\delta(\cdot,u)}V_0(\bar{x})$; i.e., $\Delta_k V_0(x_k)\leq\delta u_k H^\delta_{av}(\bar{x}_k,u_k)$ for any $k\geq 0$. The state-feedback implicitly defined by (20) ensures

$$\Delta_k V_0(\bar{x}_k)\leq-\delta K\|H^\delta_{av}(\bar{x}_k,u_k)\|^2\qquad(21)$$

so that GAS of the equilibrium follows whenever Σ^δ with output $\mathrm{L}_{\bar{G}^\delta(\cdot,0)}V_0(\bar{x})$ is ZSD. This is inherited from the ZSD property of Σ_c with output $\mathrm{L}_{\bar{g}}V_0(z,\xi)$ and because by construction $\mathrm{L}_{\bar{G}^\delta(\cdot,0)}V_0(\bar{x})=\delta\mathrm{L}_{\bar{g}}V_0(z,\xi)+O(\delta^2)$.

Again, the controller u^{av} is in the form (8) with first terms given by

$$\begin{aligned}u^{av}_0=&u_c\big|_{t=k\delta},\quad u^{av}_1=\dot{u}_c\big|_{t=k\delta}-L_{\bar{g}}L_{\bar{f}}V_0(\bar{x})\Big|_{\bar{x}_k}\\ u^{av}_2=&\ddot{u}_c\big|_{t=k\delta}-\frac{1}{2}\dot{u}_c\big|_{t=k\delta}L^2_{\bar{g}}V_0(\bar{x}_k)\\ &-(L_{\bar{f}}L_{\bar{g}}L_{\bar{f}}+L_{\bar{g}}L^2_{\bar{f}}-u_c\big|_{t=k\delta}L^2_{\bar{g}}L_{\bar{f}})V_0(\bar{x}_k).\end{aligned}$$

We note that the design based on u-average passivity lies in a purely discrete-time framework.

Remark 4.4: When comparing the two feedback u^{av} and u^δ in (20)and (14) one gets that they differ for the term

$$\begin{aligned}&u^{av}-u^\delta\\ =&(T^\delta_1(\bar{x},u^{av}))^{-1}K\int_0^\delta\int_0^1 L_{\bar{G}^s(\cdot,\theta u^{av})}L_{\bar{f}}V(\bar{x}^+(\theta u^{av}))d\theta ds\end{aligned}\qquad(22)$$

with T^δ_1 as in (17). Thus, these two strategies yield the same solution when Σ_c is lossless. In this case, Σ^δ is u-average lossless.

Example (cont'd): When considering (9), the u-average output mapping is provided by

$$\begin{aligned}\delta H^\delta_{av}(z,\xi,u)=&\frac{u}{2}+\delta z+\delta\xi+\frac{\delta^2}{2}u+\frac{\delta}{3}\xi^3-ue^{-\delta}+\\ &\frac{1}{2}e^{-2\delta}u+\xi e^{-\delta}-\xi e^{-2\delta}.\end{aligned}$$

and the expression of u^{av} is finitely computable and it gets the form:

$$u^{av}=-D^{av}(\delta)(3\xi e^\delta-3\xi+3\delta ze^{2\delta}+e\delta\xi e^{2\delta}+\delta\xi^3e^{2\delta})$$

with $D^{av}(\delta)=\frac{1}{6}(e^{2\delta}-2e^\delta+\delta^2e^{2\delta}+2\delta e^{2\delta}+1)^{-1}$.

V. SIMULATIONS

Simulations are referred to the dynamics (9) by implementing the CT control, the three proposed control strategies and the emulated based one (i.e., when the continuous-time controller is implemented by means of Zero-Order-Holder devices). Figures 1 to 3 depict the results for initial state $\bar{x}=(0,0)^\top$ and increasing values of the sampling period. We plot the output $y_c=L_{\bar{g}}V_0$. We see that the three proposed control strategies yield more than acceptable performances even when the emulated-based control fails. We point out, that though the δ and u-average controllers are exactly computed, only the second order approximate solution is implemented for the I-LM control.

VI. CONCLUDING COMMENTS

We discussed the sampled-data stabilization of a two-block cascade feedforward dynamics through three control strategies. We showed that if Σ_c admits a CT-PBC control with storage function V_0, the SD design can be carried out without requiring further assumptions. While the discussed approaches follow different passivity based concepts, they all share the following properties.

- u^δ, u^{ilm} and u^{av} ensure closed-loop stability of the equilibrium of Σ^δ (equivalently of Σ_c under SD controller) with the same storage function as in continuous time.

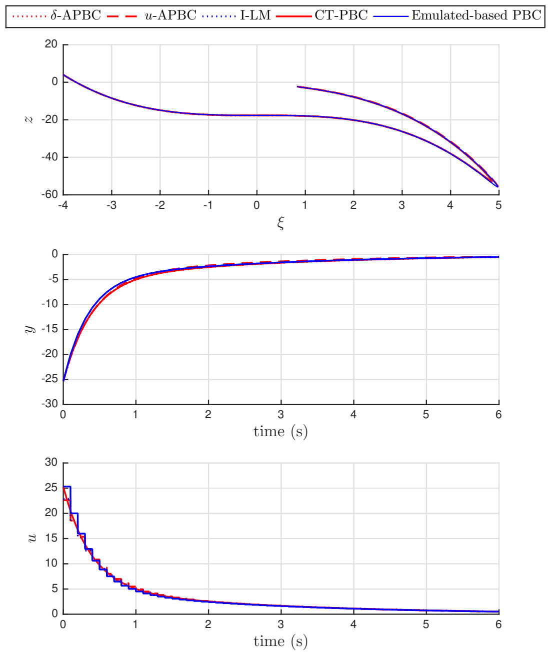

Fig. 1. $\delta = 0.1$ s.

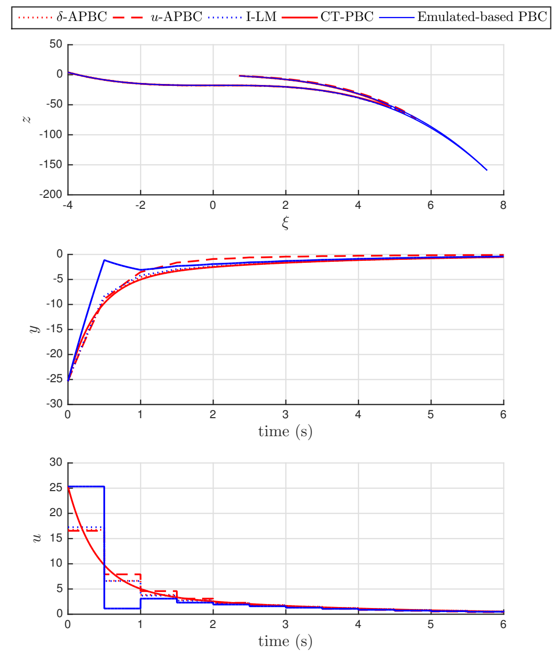

Fig. 2. $\delta = 0.5$ s.

- The SD design is constructive. An executable algorithm can be implemented for computing the solutions in the form (8).
- Since u^{ilm}, u^{δ} and u^{av} are implicitly characterized, respectively, by (10), (14) and (20). Approximate solutions can be defined as truncations of (8) at finite order p so achieving GAS in a practical sense [4].
- Alternative bounded solutions can be implemented based on the first order computable approximate solutions. As discussed [12] and [13], those feedbacks still guarantee global asymptotic stabilization.

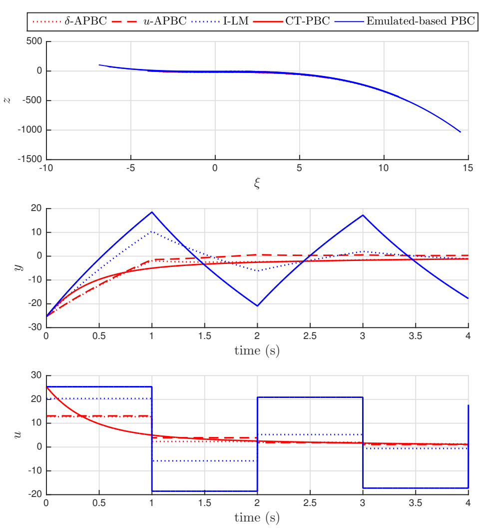

Fig. 3. $\delta = 1$ s.

REFERENCES

[1] M. Malisoff and F. Mazenc, *Constructions of strict Lyapunov functions*. Springer Science & Business Media, 2009.

[2] R. Sepulchre, M. Janković, and P. Kokotović, *Constructive nonlinear control*. Springer New York, 1997.

[3] L. Praly, R. Ortega, and G. Kaliora, "Stabilization of nonlinear systems via forwarding mod {L g V}," *Automatic Control, IEEE Transactions on*, vol. 46, no. 9, pp. 1461–1466, 2001.

[4] V. Tanasa, S. Monaco, and D. Normand-Cyrot, "Backstepping control under multi-rate sampling," *IEEE Transactions on Automatic Control*, May, 2016.

[5] S. Monaco, D. Normand-Cyrot, and F. Tiefensee, "Sampled-data stabilization; a pbc approach," *IEEE Transactions on Automatic Control*, vol. 56, pp. 907–912, 2011.

[6] R. Costa-Castelló and E. Fossas, "On preserving passivity in sampled-data linear systems," *European Journal of Control*, vol. 13, no. 6, pp. 583–590, 2007.

[7] S. Monaco and D. Normand-Cyrot, "Nonlinear average passivity and stabilizing controllers in discrete-time," *Systems & Control Letters*, vol. 60, pp. 431–439, 2011.

[8] ——, "On the differential/difference representation of sampled dynamics," in *IEEE Conference on Decision and Control*, vol. 44, no. 7, 2005, pp. 6597 – 6602.

[9] ——, "Issues on nonlinear digital systems," *Semi-Plenary Conference, European Journal of Control*, vol. 7, no. 2-3, pp. 160–178, 2001.

[10] F. Tiefensee, S. Monaco, and D. Normand-Cyrot, "Ida-pbc under sampling for port-controlled Hamiltonian systems," in *American Control Conference (ACC)*. IEEE, 2010, pp. 1811–1816.

[11] S. Stramigioli, C. Secchi, A. J. Van Der Schaft, and C. Fantuzzi, "Sampled data systems passivity and discrete port-hamiltonian systems," *IEEE Transactions on Robotics*, vol. 21, no. 4, pp. 574–587, 2005.

[12] F. Mazenc and H. Nijmeijer, "Forwarding in discrete-time nonlinear systems," *International Journal of Control*, vol. 71, no. 5, pp. 823–835, 1998.

[13] S. Monaco, D. Normand-Cyrot, and M. Mattioni, "Stabilization of feedforward discrete-time dynamics through immersion and invariance," in *American Control Conference*, Boston (USA), 2016.